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BOUNDARY-DOMAIN ELEMENT METHOD FOR FREE VIBRATION OF MODERATELY THICK LAMINATED ORTHOTROPIC SHALLOW SHELLS

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Abstract— In this paper an integral equation formulation of free vibration of moderately thick laminated orthotropic shallow shells has been established by using the method of weighted residuals and taking the static fundamental solution as the weighted function. In the present formulation there are not only boundary unknowns but also domain unknowns. The integral equations presented are reduced to a standard algebraic eigenvalue problem by means of the boundary-domain element. In the numerical implementation, two kinds of different matrix formulation are presented by using two kinds of different domain elements. Some numerical results are presented and compared with the exact solutions in order to demonstrate the correctness and the accuracy of the present method.

1. INTRODUCTION

During the last few decades, boundary integral equation methods have been widely applied in the analysis of structure engineering. Much research work has been done on free vibration of plates and shells. There are basically two approaches for treating free vibration of plates and shells by using boundary integral equation methods. The first approach is that the fundamental solutions which are dependent on frequency are taken as the weighted function. This approach reduces a non-algebraic eigenvalue problem, that is to say, the matrix solving eigenvalue is dependent on frequency (Vivoli and Filippi, 1974; Niwa *et al.*, 1981; Kitahara, 1985; Wong and Hutchinson, 1981). The second approach is that the static fundamental solutions are taken as the weighted function. This approach reduces not only boundary unknowns but domain unknowns in the resulting integral equation. This integral equation is solved by boundary-domain element methods. The great advantage of this approach is that it yields a standard algebraic eigenvalue problem. This method was applied to thin plate vibration (Bezine, 1980; Costa, 1988; Providakis and Beskos, 1989). It was also applied to Hoff's sandwich plate vibration (Wang and Huang, 1992a) and orthotropic thick plate vibration (Wang and Huang, 1992b). Boundary-domain element methods were also applied to the static analysis of shells (Ye, 1991); however, to the authors' knowledge, boundary-domain element methods have not been used in free vibration of moderately thick laminated orthotropic shallow shells.

In this paper the static fundamental solution is taken as the weighted function. An integral formulation for free vibration of moderately thick laminated orthotropic shallow shells has been established. In the present formulation, there are not only boundary unknowns but also domain unknowns. The integral equation formulation presented is reduced to a standard algebraic eigenvalue problem by the boundary-domain element. Some examples are analysed with the present method and compared with the exact solution. Numerical results show that the method presented in this paper has a good accuracy and a high efficiency.

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2. INTEGRAL EQUATION FORMULATION

In this paper we shall only consider orthotropic and symmetric cross-ply laminated shallow shells. Herein, the plane of x_1 and x_2 is assumed to coincide with the midsurface of the shallow shells with constant thickness h. The principal axes of orthotropy are assumed to coincide with x_1 and x_2. U_i represents the displacements of the shallow shells in the directions of x_1, x_2 and x_3, and the rotations in the directions of x_1 and x_2, i.e. U_i indicates u, v, w, ψ_x and ψ_y. According to the simplified theory of shallow shells, the basic equations of free vibration of shallow shells including transverse shear deformation can be expressed in the following form (Wang, 1991):

$$\Delta^*_{ij}U_j+\omega^2 m_{ij}U_j = 0 \quad (i,j = 1,2,3,4,5). \tag{1}$$

where ω is the circular frequency of the shells, Δ^*_{ij} are differential operators and m_{ij} is a diagonal matrix, which can be expressed as follows:

$$\Delta^*_{11} = C_{11}\frac{\partial^2}{\partial x_1^2}+C_{66}\frac{\partial^2}{\partial x_2^2} \quad \Delta^*_{22} = C_{66}\frac{\partial^2}{\partial x_1^2}+C_{22}\frac{\partial^2}{\partial x_2^2}$$

$$\Delta^*_{33} = \nabla_c^2-k_1^2C_{11}-2k_1k_2C_{12}-k_2^2C_{22} \quad \Delta^*_{44} = D_{11}\frac{\partial^2}{\partial x_1^2}+D_{66}\frac{\partial^2}{\partial x_2^2}-C_1$$

$$\Delta^*_{55} = D_{66}\frac{\partial^2}{\partial x_1^2}+D_{22}\frac{\partial^2}{\partial x_2^2}-C_2 \quad \Delta^*_{12} = \Delta^*_{21} = (C_{12}+C_{66})\frac{\partial^2}{\partial x_1\,\partial x_2}$$

$$\Delta^*_{13} = -\Delta^*_{31} = (C_{11}k_1+C_{12}k_2)\frac{\partial}{\partial x_1} \quad \Delta^*_{23} = -\Delta^*_{32} = (C_{12}k_1+C_{22}k_2)\frac{\partial}{\partial x_2}$$

$$\Delta^*_{34} = -\Delta^*_{43} = C_1\frac{\partial}{\partial x_1} \quad \Delta^*_{35} = -\Delta^*_{53} = C_2\frac{\partial}{\partial x_2}$$

$$\Delta^*_{45} = \Delta^*_{54} = (D_{12}+D_{66})\frac{\partial^2}{\partial x_1\,\partial x_2}$$

$$\Delta^*_{14} = \Delta^*_{41} = \Delta^*_{15} = \Delta^*_{51} = \Delta^*_{24} = \Delta^*_{42} = \Delta^*_{25} = \Delta^*_{52} = 0$$

$$\nabla_c^2 = C_1\frac{\partial^2}{\partial x_1^2}+C_2\frac{\partial^2}{\partial x_2^2} \quad C_1 = K_1^2C_{55} \quad C_2 = K_2^2C_{44}$$

$$C_{ij} = \sum_{k=1}^{n}(\bar{Q}_{ij})_k(h_k-h_{k-1}) \quad D_{ij} = \frac{1}{3}\sum_{k=1}^{n}(\bar{Q}_{ij})_k(h_k^3-h_{k-1}^3)$$

$$m_{11} = m_{22} = m_{33} = \bar{\rho} = \sum_{k=1}^{n}\rho^{(k)}(h_k-h_{k-1})$$

$$m_{44} = m_{55} = J = \frac{1}{3}\sum_{k=1}^{n}\rho^{(k)}(h_k^3-h_{k-1}^3). \tag{2}$$

where k_1 and k_2 are two principal curvatures of shallow shells, the expression $\bar{Q}_{ij}$ can be found in Wang (1991) and Vinson and Chou (1975), h_k is the vertical distance from the midplane, $z = 0$, to the upper surface of the kth lamina, $\rho^{(k)}$ is the density of the kth lamina and K_1 and K_2 are the shear correction factors (Wang, 1991; Vinson and Chou, 1975).

By using the method of weighted residuals, eqn (1) can be expressed in the following form:

$$-\int_{\Omega}(\Delta^*_{kj}U^*_{ij}+\omega^2U^*_{ij}m_{jk})U_k\,d\Omega=\int_{\Gamma}U^*_{ij}P_j\,d\Gamma-\int_{\Gamma}P^*_{ij}U_j\,d\Gamma\quad(i,j=1,2,3,4,5). \tag{3}$$

In the above equation, if we take

$$\Delta^*_{kj}U^*_{ij}+\omega^2U^*_{ij}m_{jk}=-\delta_{ki}\delta(\zeta,\mathbf{x})\quad(i,j=1,2,3,4,5), \tag{4}$$

in which $\delta(\zeta,\mathbf{x})$ is the Dirac delta function, then the fundamental solution $U^*_{ij}(\zeta,\mathbf{x})$ obtained by using eqn (4) is dependent on frequency. In this paper the static fundamental solution of the problem is taken as the weighted function, i.e. we take

$$\Delta^*_{kj}U^*_{ij}(\zeta,\mathbf{x})=-\delta_{ki}\delta(\zeta,\mathbf{x})\quad(i,j=1,2,3,4,5). \tag{5}$$

Substituting eqn (5) into eqn (3), we have

$$\begin{aligned}C_{ij}(\zeta)U_j(\zeta)&=\int_{\Gamma}U^*_{ij}(\zeta,\mathbf{x})P_j(\mathbf{x})\,d\Gamma-\int_{\Gamma}P^*_{ij}(\zeta,\mathbf{x})U_j(\mathbf{x})\,d\Gamma\\&\quad+\omega^2\int_{\Omega}(\bar{\rho}U^*_{i1}(\zeta,\mathbf{x})U_1(\mathbf{x})+\bar{\rho}U^*_{i2}(\zeta,\mathbf{x})U_2(\mathbf{x})+\bar{\rho}U^*_{i3}(\zeta,\mathbf{x})U_3(\mathbf{x})\\&\qquad+JU^*_{i4}(\zeta,\mathbf{x})U_4(\mathbf{x})+JU^*_{i5}(\zeta,\mathbf{x})U_5(\mathbf{x}))\,d\Omega\quad(i,j=1,2,3,4,5),\end{aligned} \tag{6}$$

where $U^*_{ij}(\zeta,\mathbf{x})$ and $P^*_{ij}(\zeta,\mathbf{x})$ are the static fundamental solutions of the generalized displacements and the generalized boundary forces of moderately thick laminated orthotropic shallow shells (Wang and Huang, 1994; Wang and Schweizerhof, 1995). ζ and $\mathbf{x}$ represent the coordinates of the source point and a field point, respectively. $C_{ij}(\zeta)$ depends on the position of ζ. If $\zeta\in\Omega$, then $C_{ij}=\delta_{ij}$; if ζ is on the smooth boundary, then $C_{ij}=\delta_{ij}/2$; and if ζ is on the non-smooth boundary, C_{ij} depends on the geometry of the boundary. In general, if ζ is on the non-smooth boundary, it is difficult to obtain $C_{ij}(\zeta)$ analytically, but $C_{ij}(\zeta)$ can be obtained by the indirect method (Vander Weeën, 1982) from the point of view of numerical calculation. On the other hand, as ζ is artificially selected, the corner point can be avoided.

3. MATRIX FORMULATION

In general, it is difficult to obtain the analytical solution of integral eqn (6). Thus, eqn (6) is solved by using a numerical method. As there are not only boundary unknowns but also domain unknowns in eqn (6), the boundary Γ and the domain Ω are both discretized into a number of boundary and domain elements. A matrix formulation of the algebraic eigenvalue for free vibration of moderately thick laminated orthotropic shallow shells will be established. In the procedure of numerical implementation, the boundary is discretized into N_e boundary elements with N_p nodes. The generalized boundary displacements and the generalized boundary forces are interpolated in terms of their nodal values on every boundary element. Boundary elements can be constant or higher order elements. The domain is discretized into M domain elements. When a different interpolation function is employed in every domain element, a different matrix formulation is obtained. When the generalized displacements are taken as constant in every domain element, the third term on the right-hand side of eqn (6) is only related to the generalized displacements of domain nodes. When a higher order interpolation function is employed in every domain element, the third term on the right-hand side of eqn (6) is related not only to the generalized displacements of domain nodes but also to the generalized displacement of boundary nodes. In what follows, we shall consider the two cases in which constant elements and higher order elements are used in the domain.

3.1. *Using constant elements in the domain*

As there are not only boundary unknowns but also domain unknowns in eqn (6), the boundary and the domain of the shell are divided into a number of boundary and domain elements. The generalized displacements of the shell are taken as constant in every domain element. Boundary elements can be constant or higher order elements. We shall establish integral eqn (6) on every boundary node and at the centre of every domain element.

(a) When $\zeta \in \Gamma$, i.e. the source point ζ is situated at every boundary node, by using eqn (6) successively on all the nodal points of the boundary and considering the boundary conditions, we have

$$[\mathbf{A}_{\Gamma\Gamma}]\{\mathbf{X}\} - \omega^2[\mathbf{B}_{\Gamma\Omega}]\{\mathbf{U}_\Omega\} = \mathbf{0}, \tag{7}$$

where $[\mathbf{A}_{\Gamma\Gamma}]$ and $[\mathbf{B}_{\Gamma\Omega}]$ are the coefficient matrices, $\{\mathbf{X}\}$ are the nodal unknowns of the generalized tractions and the generalized displacements on the boundary and $\{\mathbf{U}_\Omega\}$ are the unknowns of the generalized displacements at the centre of every domain element.

(b) When $\zeta \in \Omega$, i.e. the source point ζ is situated at the centre of every domain element, by using eqn (6) once again on all the domain elements and considering the boundary conditions, we have

$$\{\mathbf{U}_\Omega\} + [\mathbf{A}_{\Omega\Gamma}]\{\mathbf{X}\} = \omega^2[\mathbf{B}_{\Omega\Omega}]\{\mathbf{U}_\Omega\}, \tag{8}$$

in which $[\mathbf{A}_{\Omega\Gamma}]$ and $[\mathbf{B}_{\Omega\Omega}]$ are the coefficient matrices. $\{\mathbf{X}\}$ can be obtained by using eqn (7), which is

$$\{\mathbf{X}\} = \omega^2[\mathbf{A}_{\Gamma\Gamma}]^{-1}[\mathbf{B}_{\Gamma\Omega}]\{\mathbf{U}_\Omega\}. \tag{9}$$

Substituting the above expression into eqn (8), we obtain the following standard matrix formulation of the algebraic eigenvalue problem:

$$\left([\mathbf{M}] - \frac{1}{\omega^2}[\mathbf{I}]\right)\{\mathbf{U}_\Omega\} = 0, \tag{10}$$

in which $[\mathbf{I}]$ is a unit matrix and

$$[\mathbf{M}] = [\mathbf{B}_{\Omega\Omega}] - [\mathbf{A}_{\Omega\Gamma}][\mathbf{A}_{\Gamma\Gamma}]^{-1}[\mathbf{B}_{\Gamma\Omega}]. \tag{11}$$

3.2. *Using higher order elements in the domain*

In this discretization scheme, higher order elements, which may be linear elements, quadratic elements or cubic elements, are employed in the domain. Boundary elements can be constant or higher elements. As there are not only boundary unknowns but also domain unknowns in eqn (6), we shall establish integral equation (6) at every boundary node and on every domain node.

(a) When $\zeta \in \Gamma$, i.e. the source point ζ is situated at every boundary node, by using eqn (6) successively on all the boundary nodes and considering the boundary conditions, we have

$$[\mathbf{A}_{\Gamma\Gamma 1} \vdots \mathbf{A}_{\Gamma\Gamma 2}]\begin{Bmatrix}\mathbf{U}_\Gamma \\ \mathbf{X}\end{Bmatrix} - \omega^2[\mathbf{B}_{\Gamma\Omega}]\begin{Bmatrix}\mathbf{U}_\Gamma \\ \mathbf{U}_\Omega\end{Bmatrix} = \mathbf{0}, \tag{12}$$

in which $[\mathbf{A}_{\Gamma\Gamma 1}]$, $[\mathbf{A}_{\Gamma\Gamma 2}]$ and $[\mathbf{B}_{\Gamma\Omega}]$ are the coefficient matrices. $\{\mathbf{U}_\Gamma\}$ and $\{\mathbf{U}_\Omega\}$ are the nodal unknowns of the generalized displacements on the boundary and in the domain, respectively. $\{\mathbf{X}\}$ represents the nodal unknowns of the generalized tractions on the boundary.

(b) When $\zeta \in \Omega$, i.e. the source point ζ is situated at every node of the domain, by using eqn (6) successively to all the domain nodes and considering the boundary conditions, we have

$$\{U_\Omega\} + [A_{\Omega\Gamma 1} \vdots A_{\Omega\Gamma 2}]\begin{Bmatrix} U_\Gamma \\ X \end{Bmatrix} - \omega^2 [B_{\Omega\Omega}]\begin{Bmatrix} U_\Gamma \\ U_\Omega \end{Bmatrix} = 0, \tag{13}$$

in which $[A_{\Omega\Gamma 1}]$, $[A_{\Omega\Gamma 2}]$ and $[B_{\Omega\Omega}]$ are the coefficient matrices.

Equations (12) and (13) can be expressed in the following form:

$$[A_{\Gamma\Gamma 1}]\{U_\Gamma\} + [A_{\Gamma\Gamma 2}]\{X\} - \omega^2 [B_{\Gamma\Omega}]\begin{Bmatrix} U_\Gamma \\ U_\Omega \end{Bmatrix} = 0 \tag{14}$$

$$\{U_\Omega\} + [A_{\Omega\Gamma 1}]\{U_\Gamma\} + [A_{\Omega\Gamma 2}]\{X\} - \omega^2 [B_{\Omega\Omega}]\begin{Bmatrix} U_\Gamma \\ U_\Omega \end{Bmatrix} = 0. \tag{15}$$

Using eqn (14), we obtain

$$\{X\} = -[A_{\Gamma\Gamma 2}]^{-1}[A_{\Gamma\Gamma 1}]\{U_\Gamma\} + \omega^2 [A_{\Gamma\Gamma 2}]^{-1}[B_{\Gamma\Omega}]\begin{Bmatrix} U_\Gamma \\ U_\Omega \end{Bmatrix}. \tag{16}$$

Substituting eqn (16) into eqn (15), we obtain the following matrix formulation of standard algebraic eigenvalue problem:

$$\left([M] - \frac{1}{\omega^2}[K]\right)\{U\} = 0, \tag{17}$$

in which

$$[M] = [B_{\Omega\Omega}] - [A_{\Omega\Gamma 2}][A_{\Gamma\Gamma 2}]^{-1}[B_{\Gamma\Omega}]$$

$$[K] = [([A_{\Omega\Gamma 1}] - [A_{\Omega\Gamma 2}][A_{\Gamma\Gamma 2}]^{-1}[A_{\Gamma\Gamma 1}]) \vdots [I]]$$

$$\{U\} = \begin{Bmatrix} U_\Gamma \\ U_\Omega \end{Bmatrix}, \tag{18}$$

where $[M]$ and $[K]$ are known. Equations (10) and (17) describe an algebraic eigenvalue problem of a non-symmetric matrix. The natural frequencies and mode shapes can be obtained from the eigenvalues and eigenvectors provided by the solution of eqns (10) or (17).

4. NUMERICAL RESULTS

In this section some numerical results are given to verify the correctness of the formulation and the accuracy of the solution which are presented in this paper. R_1 and R_2 are two principal radii of the shell along the x- and y-axes, respectively. a and b are the lengths of the shell along the x- and y-axes, respectively. Vibrational mode shapes are usually described in the form m, n, where m and n are the numbers of half waves in the x and y directions, respectively. The entire boundary of the shell is divided equally into 32 boundary elements and a linear interpolation function is employed on every boundary element. The domain is divided equally into two kinds of 4×4 (16 domain elements) and 8×8 meshes (64 domain elements). The generalized displacements of the shell are taken as constant on every domain element, i.e. constant elements are used in the domain of the shell. The exact

Table 1. Frequency parameters [$\bar{\omega} = \omega a^2\sqrt{(\rho/E_2)}/h$] of SSSS moderately thick laminated orthotropic square spherical shallow shell for the case of 0°/90°/90°/0°

R/a	h/a	Method	Wave number (m, n) 1,1	1,2	2,1	2,2	1,3
3	0.01	4 × 4	48.82257	65.17546	81.77706	82.91255	93.63143
		8 × 8	47.95226	62.66462	78.29612	77.97635	90.59067
		Reddy (1984)	47.41525	60.97662	76.26018	75.13405	86.46918
	0.1	4 × 4	13.31602	24.27564	31.71891	38.44838	41.75744
		8 × 8	13.05551	23.20745	30.39797	36.29395	39.79291
		Reddy (1984)	12.79278	22.63980	29.72045	35.17592	38.31140
5	0.01	4 × 4	32.06317	46.06915	67.41534	72.99397	74.45932
		8 × 8	31.46153	44.09002	64.44392	68.43393	70.80062
		Reddy (1984)	31.07907	42.92415	62.92941	65.99475	67.79810
	0.1	4 × 4	12.86442	23.89799	31.47222	38.29665	41.42246
		8 × 8	12.61308	22.84572	30.16340	36.15446	39.46570
		Reddy (1984)	12.43619	22.36235	29.56296	35.10534	38.07252
10	0.01	4 × 4	21.05866	34.69995	60.18495	68.27831	64.18169
		8 × 8	20.65688	33.12916	57.49284	63.92740	60.40510
		Reddy (1984)	20.38024	32.30546	56.30135	61.71514	58.08370
	0.1	4 × 4	12.66091	23.73201	31.36421	38.22723	41.27410
		8 × 8	12.41384	22.68744	30.06171	36.09381	39.32355
		Reddy (1984)	12.28005	22.24316	29.49562	35.07542	37.97058

solution of the simply supported shell is calculated by using the formulation [eqn (27)] presented by Reddy (1984) and not taken directly from the paper of Reddy (1984). Except where indicated to the contrary, the values of the shear correction factors are $K_1^2 = K_2^2 = 5/6$.

Example 1

A simply supported (SSSS) spherical shallow shell is considered. The shell is a four-layer symmetric orthotropic cross-ply laminate, i.e. 0°/90°/90°/0°. The material properties for all piles are identical. The following geometric and material parameters are used:

$$a/b = 1 \quad R_1 = R_2 = R$$
$$E_1 = 25E_2 \quad G_{23} = 0.2E_2$$
$$G_{13} = G_{12} = 0.5E_2 \quad \nu_{12} = 0.25.$$

Table 1 shows a comparison between the present method and the exact solution (Reddy, 1984).

Example 2

A simply supported (SSSS) non-spherical shallow shell is considered. The material parameters and laminate case of the shell are the same as in example 1. The following two kinds of geometric parameters are used:

$$\text{case 1} \quad a/b = 1 \quad R_1/a = 8 \quad R_2/a = 10$$
$$\text{case 2} \quad a/b = 1 \quad R_1/a = 8 \quad R_2/a = 16.$$

A comparison between the present solution and the exact solution (Reddy, 1984) is given in Table 2.

Example 3

A square spherical shallow shell whose two opposite edges parallel to the y-axis are simply supported and the other edges clamped is considered. This kind of boundary condition is represented by SCSC. The shell is a five-layer orthotropic cross-ply laminate,

Table 2. Frequency parameters [$\bar{\omega} = \omega a^2\sqrt{(\rho/E_2)}/h$] of SSSS moderately thick laminated orthotropic square non-spherical shallow shell for the case of 0°/90°/90°/0°

			Wave number (m, n)				
R/a	h/a	Method	1,1	1,2	2,1	2,2	1,3
$R_1/a = 8$	0.01	4×4	22.25957	36.62644	60.41892	68.70238	66.15901
$R_2/a = 10$		8×8	21.83573	34.98081	57.71851	64.32666	62.37116
		Reddy (1984)	21.54605	34.07575	56.53498	62.10343	59.81966
	0.1	4×4	12.67941	23.75739	31.36784	38.23435	41.29947
		8×8	12.43193	22.71151	30.06471	36.09933	39.34791
		Reddy (1984)	12.29182	22.26177	29.49453	35.07687	37.98934
$R_1 = 8$	0.01	4×4	20.47568	35.89838	59.00336	68.07171	65.84282
$R_2/a = 16$		8×8	20.08482	34.28034	56.35698	63.72138	62.05236
		Reddy (1984)	19.81710	33.38576	55.24961	61.52753	59.51194
	0.1	4×4	12.65103	23.74758	31.34670	38.22346	41.29308
		8×8	12.40417	22.70223	30.04478	36.09073	39.34335
		Reddy (1984)	12.26740	22.25491	29.47750	35.07048	37.98743

Table 3. Fundamental frequency parameters [$\bar{\omega} = \omega a^2\sqrt{(\rho/E_T)}/h$] of SCSC moderately thick laminated orthotropic square spherical shallow shell for the case of 0°/90°/0°/90°/0°, $h/a = 0.01$

R/a	10	20	30	40	50	60	Plate
4×4	42.95384	30.54981	27.60329	26.49045	25.95817	25.66396	(Craig and Dawe, 1986)
8×8	42.04798	29.88533	26.99901	25.90925	25.38809	25.10009	24.3717

Table 4. Frequency parameters [$\bar{\omega} = \omega a^2\sqrt{(\rho/E_T)}/h$] of SCSC moderately thick laminated orthotropic square spherical shallow shell for the case of 0°/90°/0°/90°/0°

			Wave number (m, n)				
h/a	R/a	Method	1,1	1,2	2,1	2,2	1,3
0.01	10	4×4	42.95384	66.35318	70.83711	90.23179	138.8637
		8×8	42.04798	62.25563	67.53970	82.94850	113.5904
	20	4×4	30.54981	63.88027	63.74912	88.73326	135.9503
		8×8	29.88533	59.91585	60.71142	81.52518	110.9735
Plate	(Craig and Dawe, 1986)		24.3717	56.6809	57.1459	78.1717	107.072
0.1	10	4×4	18.28430	35.13094	39.41846	50.97265	53.14729
		8×8	17.78768	32.94824	36.98088	46.36815	49.90220
	20	4×4	18.13904	35.62283	40.02329	53.02312	53.88698
		8×8	17.54516	32.93322	36.87080	46.79841	50.09028
Plate	(Craig and Dawe, 1986)		17.6916	31.1819	36.3093	44.6448	48.0932

i.e. 0°/90°/0°/90°/0°, whose material properties for all plies are identical. The material properties and geometric parameters are as follows:

$$a/b = 1 \quad R_1 = R_2 = R \quad E_L/E_T = 30$$

$$G_{LT}/E_T = 0.6 \quad G_{TT}/E_T = 0.5 \quad \nu_{LT} = 0.25,$$

where subscripts L and T refer to directions parallel to the fibres and transverse to the fibres, respectively. The thickness of each of the 0° plies is two-thirds that of each of the 90° plies, so that the total sum of the thicknesses of the 0° and 90° plies is the same. The shear correction factors are $K_1^2 = 0.87323$ and $K_2^2 = 0.59139$. The data used in this example are taken from Craig and Dawe (1986) and Wang and Dawe (1993). The numerical results are given in Tables 3 and 4 in terms of a frequency parameter $\bar{\omega}$.

We see from Tables 1 and 2 that the natural frequencies obtained by using the simplest domain element (constant element) in the present method are in excellent agreement with the exact solution (Reddy, 1984). When the domain is divided into an 8×8 mesh, the maximum error of the natural frequencies is not greater than 5% of the exact solution (Reddy, 1984). We see from Tables 3 and 4 that the solution of the spherical shell gradually tends to the solution of the plate with increasing curvature radius R of the shell. This is in keeping with the actual situation.

5 CONCLUSIONS

In this paper an integral equation formulation for free vibration of moderately thick laminated orthotropic shallow shells has been presented by using the static fundamental solution as the weighted function. As there are not only boundary unknowns but also domain unknowns in the present integral equation, the boundary and the domain are discretized. The boundary-domain element method for free vibration of moderately thick orthotropic laminated shallow shells is presented in which the resulting integral equations presented are reduced to a standard matrix eigenvalue problem. In this paper, two kinds of different matrix formulation are presented by using two kinds of different domain elements. In the present method, the domain discretization scheme required to solve the problem using the present method is much simpler than those necessary with the finite element method. Using the present method a more accurate result can be obtained with a small number of boundary and domain elements. The present method can be used not only for regular region and simple boundary conditions but also for irregular region and complex boundary conditions. The numerical results show that the present method is an accurate analysis technique with good convergence for the solution of the free vibration problem of moderately thick laminated orthotropic shallow shells. The numerical calculation using higher boundary elements and higher domain elements is under study for the problem.

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